MATH4240: Stochastic Processes Tutorial 4

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Recurrent and Transient

Let X_n, $n \geq 0$, be an irreducible Markov chain with state space $\mathcal{S}$.
For $x, y \in \mathcal{S}$, let

$$G(x,y) = E_x[N(y)] = E_x[\sum_{n=1}^{\infty} 1_y(X_n)] = \sum_{n=1}^{\infty} E_x[1_y(X_n)] = \sum_{n=1}^{\infty} P^n(x,y)$$

denote the expected number of visits to y for the chain starting at x. Then

$$G(x,y) = \begin{cases} \infty, & \text{if the chain is recurrent,} \\ \dfrac{\rho_{xy}}{1-\rho_{yy}}, & \text{if the chain is transient.} \end{cases}$$

Hence the chain is recurrent if and only if the series $\sum_{n=1}^{\infty} P^n(x,y)$ is divergent.

Pólya's walk

Consider $\mathbb{Z}^d = \{(x_1, x_2, \cdots, x_d) | x_i \in \mathbb{Z}, i = 1, \cdots, d\}$, $d \geq 1$, the set of all integer points in d dimensions. A walker wanders randomly on $\mathbb{Z}^d$ starting from the origin o. At each point, he chooses with equal probability the one among the $2d$ nearest points where his next step will take him. The question is: does he always come back to the origin o?

The random walk above is called the *d-dimensional Pólya's walk*. (George Pólya, 1887-1985, a very famous Hungarian mathematician.)

Regarding the walk as an irreducible Markov chain with state space $\mathcal{S} = \mathbb{Z}^d$, the question is to check if such chains are recurrent or transient for each $d \geq 1$. Note that the walk always has period 2.

Pólya's walk

For d=1, if the walker is back to origin o at the $(2n)$th step, he has to make n to the left and n to the right. Hence

$$P^{2n}(o,o) = \frac{1}{2^{2n}}\binom{2n}{n} = \frac{(2n)!}{2^{2n}(n!)^2} \sim \frac{C_1}{\sqrt{n}},$$

where C_1 is a positive constant (independent of n). The last step follows from *Stirling's formula*:

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n,$$

where the notion $\sim$ means asymptotical equivalence, i.e.,

$$\lim_{n\to\infty} \frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}{n!} = 1.$$

Note that $\sum_{n=1}^{\infty} P^n(x,y)$ is divergent, so the chain is recurrent.

Pólya's walk

For d=2, if the walker is back to origin o at the $(2n)$th step, n steps have to go north or east. There are $\binom{2n}{n}$ possibilities to assign the n steps of these two types; the other n go south or west. For each of these choices, choose i from $\{0, 1, \cdots, n\}$, then assign i steps to go north and the other $n - i$ to go east; also assign i steps to go south and the other $n - i$ to go west. Hence

$$\begin{aligned} P^{2n}(o,o) &= \frac{1}{4^{2n}}\binom{2n}{n}\sum_{i=0}^{n}\binom{n}{i}^2 \\ &= \frac{1}{4^{2n}}\binom{2n}{n}\sum_{i=0}^{n}\binom{n}{i}\binom{n}{n-i} \\ &= \frac{1}{4^{2n}}\binom{2n}{n}^2 = \left(\frac{1}{2^{2n}}\binom{2n}{n}\right)^2 \sim \frac{C_2}{n}, \end{aligned}$$

where C_2 is a positive constant (independent of n).

Note that $\sum_{n=1}^{\infty} P^n(x,y)$ is divergent, so the chain is also recurrent.

Pólya's walk

For d=3, if the walker is back to origin o at the $(2n)$th step, n steps have to go north, east or up. Similarly we can write down the probability:

$$P^{2n}(o,o) = \frac{1}{6^{2n}}\binom{2n}{n}\sum_{i+j\le n}\left(\frac{n!}{i!j!(n-i-j)!}\right)^2.$$

Note that the term $i!j!(n-i-j)! \ge ([n/3]!)^3$. Hence

$$\begin{aligned}
P^{2n}(o,o) &\le \frac{1}{6^{2n}}\binom{2n}{n}\frac{n!}{([n/3]!)^3}\sum_{i+j\le n}\frac{n!}{i!j!(n-i-j)!} \\
&= \frac{1}{6^{2n}}\binom{2n}{n}\frac{n!}{([n/3]!)^3}3^n \sim C_3 n^{-3/2},
\end{aligned}$$

where C_3 is a positive constant (independent of n).

Note that $\sum_{n=1}^{\infty} P^n(x,y)$ is convergent, so the chain is transient (and so does every point).

Pólya's walk

Remark 1. In contrary to finite state space, we do not have any recurrent state when $d = 3$.
Remark 2. Indeed, we can show that in the case of d-dimensional Pólya's walk, $d \geq 1$,

$$P^{2n}(o, o) \sim C_d n^{-d/2}.$$

Hence the chain is transient for all $d \geq 3$.

Reducible Markov chains

Recall that a subset $\mathcal{C}$ of state space $\mathcal{S}$ is called an *irreducible closed set* if any pair x and y in $\mathcal{C}$ are communicated and $P(u, v) = 0$ for any $u \in \mathcal{C}$ and $v \in \mathcal{S} \setminus \mathcal{C}$. For any transient x, the *absorption probability* for $\mathcal{C}$ is defined as

$$\rho_{\mathcal{C}}(x) = P_x(T_{\mathcal{C}} < \infty).$$

Reducible Markov chains

In general, a finite state space $\mathcal{S}$ has the decomposition

$$\mathcal{S} = \mathcal{S}_R \cup \mathcal{S}_T = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \cdots \cup \mathcal{C}_m \cup \mathcal{S}_T,$$

where $\mathcal{S}_R$ is the collection of recurrent states in $\mathcal{S}$, $\mathcal{S}_T$ is the collection of transient states in $\mathcal{S}$, and each $\mathcal{C}_i$ is an irreducible closed set. Suppose that the transition matrix P has the following canonical form (if not, one can permute states in $\mathcal{S}$ properly):

$$P = \begin{array}{c} \begin{array}{ccccc} \mathcal{C}_1 & \mathcal{C}_2 & \cdots & \mathcal{C}_m & \mathcal{S}_T \end{array} \\ \begin{pmatrix} P_1 & 0 & \cdots & 0 & 0 \\ 0 & P_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & P_m & 0 \\ \times & \times & \cdots & \times & Q \end{pmatrix} \end{array}$$

Note that $Q^k(x, y) = P^k(x, y)$ for $x, y \in \mathcal{S}_T$. Now we show that

$$\lim_{k\to\infty} Q^k = 0$$

is the zero matrix. Indeed, for any $x, y \in \mathcal{S}_T$, as y is transient, $\sum_{k=1}^{\infty} P^k(x, y) = G(x, y) = E_x(N(y)) = \dfrac{\rho_{xy}}{1 - \rho_{yy}} < \infty$, hence we have $\lim_{k\to\infty} Q^k(x, y) = \lim_{k\to\infty} P^k(x, y) = 0$.

As a result, all eigenvalues of Q have moduli strictly less than 1. Indeed, for an eigenvalue λ of Q and a corresponding nonzero left eigenvector α, we have $\alpha Q^k = \lambda^k \alpha$ tends to zero vector as $k \to \infty$ since $\lim_{k\to\infty} Q^k = 0$. This implies $|\lambda| < 1$. Moreover, $I - Q$ is invertible since 1 is not the eigenvalue of Q.

Reducible Markov chains

In the lecture, we can use the one-step formula in matrix form to calculate the absorption probability $\rho_{\mathcal{C}_i}(x)$ for irreducible closed set $\mathcal{C}_i$ and $x \in \mathcal{S}_T$. As an example, consider the Markov chain with the following transition matrix

$$P = \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{pmatrix} 0 & 0 & 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 1/3 & 0 & 0 & 2/3 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 & 1/2 \\ 0 & 1/2 & 0 & 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 \\ 1/3 & 1/4 & 1/6 & 0 & 1/4 & 0 & 0 \end{pmatrix} \end{array}.$$

It is reducible with $\mathcal{C}_1 = \{1, 3, 6\}$, $\mathcal{C}_2 = \{2, 5\}$, $\mathcal{S}_T = \{4, 7\}$. To simplify the notions, we can regard each $\mathcal{C}_i$ as an absorbing state and define the transition probability $P(x, \mathcal{C}_i) = \sum_{y \in \mathcal{C}_i} P(x, y)$ for $x \in \mathcal{S}_T$.

Reducible Markov chains

Then the transition matrix can be written as

$$\widetilde{P} = \begin{array}{c} \begin{array}{cccc} \mathcal{C}_1 & \mathcal{C}_2 & 4 & 7 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 0 & 0 \end{pmatrix} \end{array} = \begin{pmatrix} I_2 & 0 \\ S & Q \end{pmatrix}.$$

Reducible Markov chains

Let $A = \begin{pmatrix} \rho_{\mathcal{C}_1}(4) & \rho_{\mathcal{C}_2}(4) \\ \rho_{\mathcal{C}_1}(7) & \rho_{\mathcal{C}_2}(7) \end{pmatrix}$. Then one-step formula can be written as $A = QA + S$.

Since $I - Q$ is invertible,

$$A = (I - Q)^{-1} S = \begin{pmatrix} 1/2 & -1/2 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ 1/2 & 1/2 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}. \tag{1}$$

Hence $\rho_{\mathcal{C}_1}(4) = \rho_{\mathcal{C}_2}(4) = \rho_{\mathcal{C}_1}(7) = \rho_{\mathcal{C}_2}(7) = 1/2$.

To find the limit $\lim_{k\to\infty} P^k$, we will discuss the general reducible case in the next tutorial. At least in this tutorial class we can deal with the following special case of reducible Markov chains.

Markov chains with absorbing states

If a Markov chain with n states has exactly m absorbing states, $0 < m < n$, and all other states are transient, then the transition matrix P is in the form

$$P = \begin{pmatrix} I_m & 0 \\ S & Q \end{pmatrix},$$

where 0 is the $m \times (n-m)$ zero matrix, S is a $(n-m) \times m$ matrix, and Q is a $(n-m) \times (n-m)$ matrix satisfying $Q^k \to 0_{n-m}$ as k goes to ∞.

By directed calculation,

$$\lim_{k\to\infty} P^k = \begin{pmatrix} I_m & 0 \\ A & 0 \end{pmatrix},$$

where $A = \lim_{k\to\infty}(S + QS + \cdots + Q^{k-1}S) = (I-Q)^{-1}S$.

Markov chains with absorbing states

Again we use

$$\widetilde{P} = \begin{array}{c} \begin{array}{cccc} \mathcal{C}_1 & \mathcal{C}_2 & 4 & 7 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 0 & 0 \end{pmatrix} \end{array} = \begin{pmatrix} I_2 & 0 \\ S & Q \end{pmatrix}$$

as an example. Since A has the same form as (1), we have

$$\lim_{k\to\infty} \widetilde{P}^k = \begin{array}{c} \begin{array}{cccc} \mathcal{C}_1 & \mathcal{C}_2 & 4 & 7 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{pmatrix} \end{array}.$$